§4.3 The Normal Distribution (also known as "Gaussian" or the "Bell Curve")

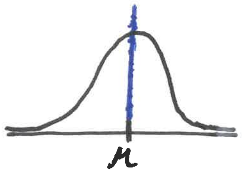
$X \sim \text{Normal}(\mu, \sigma)$

μ = mean "location"
σ = std. dev. "scale"

The normal distribution is the *most important* & widely used continuous distribution. Measurements of things which are due to many small effects acting together tend to be normal.

Examples: Person's height, weight, IQ, foot size, blood pressure...
Also measurement errors in experiments, "white noise", static....

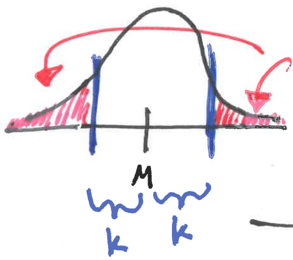
p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\left(\frac{x-\mu}{\sigma}\right)^2 \cdot \frac{1}{2}}$



↳ The c.d.f. is unpleasant and can only be written as an integral...

Note: $f(x)$ is symmetric around μ

so $P(X - \mu < -k) = P(X - \mu > k)$



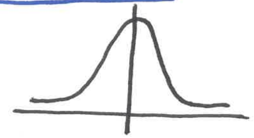
Same area = probability

Our favorite values for parameters are $\begin{cases} \mu = 0 \\ \sigma = 1 \end{cases}$

"Standard" Normal

$Z \sim \text{Normal}(0, 1)$ ("$Z \sim$ Normal")

p.d.f. $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$



Z is useful because its values tell how many std. dev. a value is from the mean.

- $Z = 3 \Longleftrightarrow 3\sigma$ from mean: very unlikely
- $Z = \frac{1}{2} \Longleftrightarrow \frac{1}{2}\sigma$ from mean: pretty likely

"Standardizing" X

We can convert between $X \sim \text{Normal}(\mu, \sigma)$ and $Z \sim \text{Normal}(0, 1)$

$$Z = \frac{X - \mu}{\sigma} \quad \text{and} \quad X = Z \cdot \sigma + \mu$$

Example: If $X \sim \text{Normal}(2, 3)$ then

- $P(X < 1) = P\left(Z < \frac{1-2}{3}\right)$
- $P(Z < -1) = P(X < (-1) \cdot 3 + 2)$

§4.3 The Normal Distribution ("Gaussian", "Bell Curve") (and lognormal)

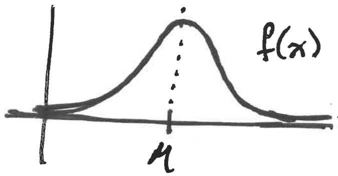
plot normal pdf in R using curve command:

```
curve(dnorm, -2, 2)
curve(dnorm(x, 2, 3), -4, 8)
```

The normal distribution is the most important and widely used continuous distribution.

$X \sim \text{Normal}(\mu, \sigma)$ $\{\mu = \text{"location"}, \sigma = \text{"scale"}\}$

p.d.f. $f(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$



The so-called "Bell curve"

People usually just keep track of standard normal because it is easy to convert to it:

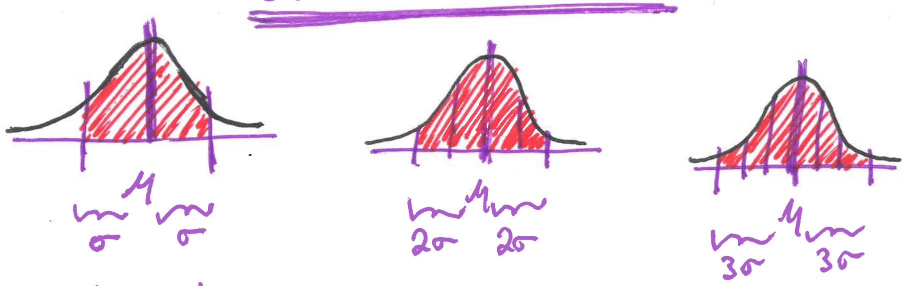
$$Z = \frac{X-\mu}{\sigma} \quad (\& \; X = \sigma Z + \mu)$$

Measurements which are the combination of many small things acting together tend to be Normal

Examples: Person's height, weight, intelligence
Measurement errors in experiments
Shoe size, blood pressure, stock returns

Normal distribution satisfies special property

68 – 95 – 99.7 rule:



$P(|X-\mu| \le \sigma)$ 68%

$P(|X-\mu| \le 2\sigma)$ 95%

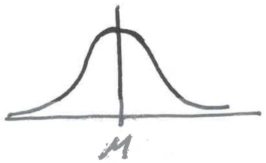
$P(|X-\mu| \le 3\sigma)$ 99.7%

Our favorite values of parameters are $\mu = 0$, $\sigma = 1$

Standard Normal

$Z \sim \text{Normal}(0, 1)$ or just "Normal"

p.d.f. $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$



The standard normal is always written with Z

Critical values z_α are values where likelihood of standard normal to be more extreme is α

$$P(Z > z_\alpha) = \alpha$$

Examples

If $X \sim \text{Normal}(2, 3)$ then compute

- $P(X < 3) \Rightarrow$ pnorm(3, 2, 3)
- $P(X > 3) \Rightarrow$ 1-pnorm(3,2,3)
- $P(0 < X < 3) \Rightarrow$ pnorm(3,2,3) - pnorm(0,2,3)
- Critical value where $P(X > x) = .05 \Rightarrow$ qnorm(1-.05, 2, 3)

Convert X to standard normal & convert all previous questions to standard normal.

$$Z = \frac{X - 2}{3}$$

$$\Rightarrow P(X < 3) = P\left(Z < \frac{3-2}{3}\right)$$

$$\Rightarrow P(X > 3) = P\left(Z > \frac{3-2}{3}\right)$$

$$\Rightarrow P(0 < X < 3) = P\left(\frac{0-2}{3} < Z < \frac{3-2}{3}\right)$$

$\boxed{\begin{array}{l}\textbf{Note: } \text{Converting } X \to Z \\ \text{"standardizing } X\text{" or "computing Z-score"} \\ \text{changes units to "\# of standard deviations from mean"}\end{array}}$

Add lognormal.
$\Rightarrow$ some things are always > 0
in this case not normal...
but often log is normal

Example IQ scores are $\text{Normal}(\mu = 100, \sigma = 15)$

{Normal Distribution is $\lim_{n\to\infty}$ Binomial Dist.}

Normal Approximation to Binomial & Poisson

If n is big and p is not too skewed then

$$\text{Binomial}(n, p) \approx \text{Normal}\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

If λ is large then

$$\text{Poisson}(\lambda) \approx \text{Normal}(\mu = \lambda, \sigma = \sqrt{\lambda})$$

§4.4 Exponential, Gamma, χ^2 ("Chi-Squared")

Recall A Poisson process is an experiment where events occur randomly at a certain rate

$$r = \frac{\#\text{ events}}{\text{length of time}}$$

Exponential

If we divide time into time periods small enough then $p = r \cdot \Delta t$ is probability of event occurring in a time period.

$$\#\begin{pmatrix}\text{time periods until} \\ \text{event occurs}\end{pmatrix} \sim \text{Geometric}\,(r \cdot \Delta t)$$

As $\Delta t \to 0$ this becomes a continuous measurement

$$\begin{pmatrix}\text{length of time} \\ \text{until event occurs}\end{pmatrix} \sim \text{Exponential}\,(r\cdot)$$

Mean: $\mu = 1/r = \dfrac{\text{length of time}}{\#\text{ of events}}$

Another point of view:

If we take a larger time period Δt then

$$\#\begin{pmatrix}\text{occurrences of event} \\ \text{in time period } \Delta t\end{pmatrix} \sim \text{Poisson}\,(r \cdot \Delta t)$$

In this case

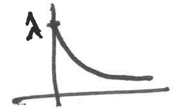
$$\begin{pmatrix}\text{length of time between} \\ \text{two occurrences}\end{pmatrix} \sim \text{Exponential}\,(r)$$

Exponential(λ)

$\lambda = \left(\begin{matrix}\text{events per} \\ \text{unit time}\end{matrix}\right)$

p.d.f.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise}\end{cases}$$



c.d.f.

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise}\end{cases}$$

!!

Exponential's mean & variance

$$\mu = 1/\lambda$$
$$\sigma^2 = 1/\lambda^2$$

Special Property: Exponential is "Memory-less"

If $X \sim \text{Exponential}\,(\lambda)$

then $P(X > x + x_0 \mid X > x_0) = P(X > x)$

Probability of waiting x more minutes if you've alredy waited x_0 minutes | Probability of waiting x minutes

WARNING: Some books will flip the parameter

$X \sim \text{Exponential}\,(\beta)$ $\quad f(x) = \frac{1}{\beta} e^{-x/\beta}$

Gamma Distribution

Exponential $\longrightarrow$ Gamma

is the continuous version of

Geometric $\longrightarrow$ Neg. Binomial.

This is the $1/\beta = \lambda$ in many people's exponential distribution

- Exponential(λ) is waiting time until event with rate λ occurs once
- Gamma (α, β) is waiting time until event with rate $= 1/\beta$ occurs α times.

$\Rightarrow$ Also defined for α not an integer !! "α = shape" "β = rate."

Uses: Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$

generalizes $n!$

$\rightarrow \Gamma(n) = (n-1)!$

$\rightarrow \Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$

$\rightarrow \Gamma(1) = 1 \qquad \Gamma(0) = 0$

$\rightarrow \Gamma(1/2) = \sqrt{\pi}$

This is used in Gamma distribution:

$X \sim \text{Gamma}(\alpha, \beta)$

p.d.f $\quad f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherw.} \end{cases}$

If $\beta = 1$ then "Standard Gamma"

$X \sim \text{Gamma}(\alpha)$

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

("Rate = 1")

(Note: If $X \sim \text{Gamma}(\alpha, \beta)$ then $X/\beta \sim \text{Gamma}(\alpha)$)

If $X \sim \text{Gamma}(\alpha, \beta)$

then $E[X] = \alpha\beta$

$\text{Var}[X] = \alpha\beta^2$

Connection between Gamma & Normal :

If $X_i \sim \text{Normal}(0, \sigma)$

then $(X_1^2 + X_2^2 + \cdots + X_k^2) \sim \text{Gamma}(k/2, 2\sigma^2)$

Chi-Squared Distribution

If $X \sim \text{Normal}(0, \sigma)$

then $X^2 \sim \text{Gamma}(1/2, 2\sigma^2)$

$$X^2/\sigma^2 \sim \text{Gamma}(1/2, 2)$$

$$\parallel$$

Chi-Squared χ^2.

More generally, $\text{Gamma}(r/2, 2)$

$$\parallel$$

$$\chi^2(r)$$

Degrees of freedom.